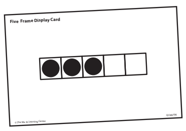
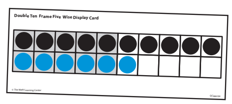
# Models in Bridges

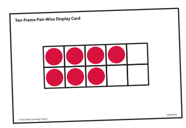
## MODELS FOR COUNTING

### Frames

K–2 students see numbers from 0 to 20 represented on five-, ten- and double-ten frames. They also use counting mats with the same frame structure to represent these numbers and compute with them. Over time, the models also help students instantly recognize quantities without having to count them.

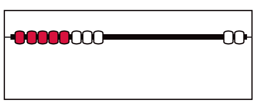

Five-Frame

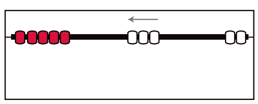

Double Ten-Frame

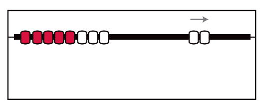

Ten-Frame

### Number Rack

Kindergartners also use the number rack to represent and compute with numbers from 0 to 10. With 5 white and 5 red beads, the number rack helps students think about groups of 5 and 10.


"1, 2, 3, 4, 5, 6, 7, 8. That's 8."

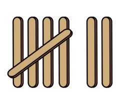

"First slide 5, and then 3 more makes 8."

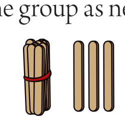

"I know it's 8 because it's 2 less than 10."

### Tallies and Bundles & Sticks

As they begin to explore the ways in which numbers can be composed and decomposed into groups, students use tallies and bundles & sticks to represent the groups of 5 and 10 in numbers greater than 5. Both tallies and bundles & sticks support students' emerging place value understandings by clearly representing the groups of 5 and 10, while also making it possible for students to count the individual lines and sticks in the group as needed.

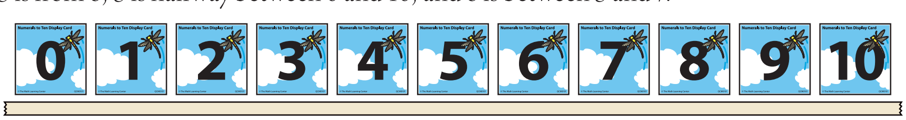

tally marks showing 8


craft sticks show 7 as tallies


a bundle of 10 and 3 more makes 13

### Number Line

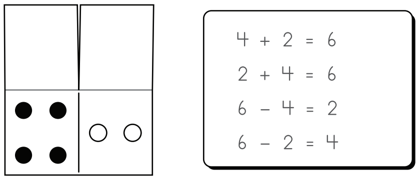
The number line helps kindergarteners connect number words with written numerals when they recite the number sequences both forward and backward. The model also helps students see relationships between numbers. For example, 1 is the same distance from 2 as 5 is from 6; 5 is halfway between 0 and 10; and 6 is between 5 and 7.



# MODELS FOR ADDITION & SUBTRACTION TO 20

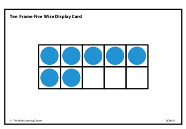
## Dominoes

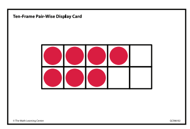
With its familiar arrangements of dots used to represent numbers, the domino is an effective tool for helping children subitize (instantly recognize quantities). Most students quickly begin to recognize the quantity represented on a domino without having to count individual dots. Dominoes are also used to model addition and subtraction.




## Ten-Frames

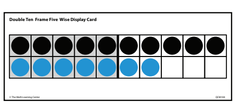
Ten-frames help students recognize quantities less than 10 without counting and help students see numbers in terms of their relationship to landmark numbers of 5 and 10, building a solid foundation for basic addition and subtraction facts within 10. Ten-frame cards are structured in 2-by-5 arrays. Dots or cubes are arranged in the array to represent numbers in either a five-wise or pairwise configuration. Five-wise frames run 5 dots or cubes across the top row before using any of the boxes in the second row. Pair-wise frames alternate dots or cubes from bottom row to top row. They encourage seeing doubles, counting by 2s, and identifying odd and even numbers.

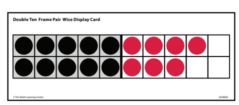

Five-wise Ten-Frames


Pair-wise Ten-Frames

## Double Ten-Frames

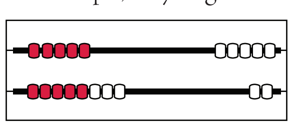
Double ten-frames help students understand teen numbers as 1 ten and some more ones since cards show one filled ten-frame and a second frame of more dots or cubes. The ten remains constant, yet accessible, and supports moving students towards unitization, seeing 10 ones as 1 ten. Since groupings of 10 are fundamental to understanding place value, this model plays an important role in developing number sense in young children.


Five-wise Double Ten-Frames


Pair-wise Double Ten-Frames

## Number Rack

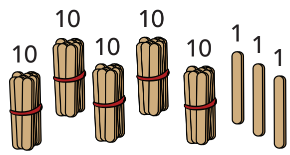
The number rack is used first as a tool to model a given number and then as a way to model and solve problems. The number rack offers visual and kinesthetic support for young learners as they begin to understand that one number can be a combination or sum of two or more other numbers. In using the number rack, students come to think of numbers as being made up of two or more distinct parts: for example, they might see 13 as 1 group of ten and 3 more.



# MODELS FOR PLACE VALUE COUNTING & EARLY COMPUTATION

## Bundles & Sticks

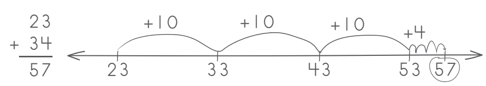
Bundles & sticks support students' emerging place value understandings by clearly representing the groups of 10, while also making it possible for students to perceive and count the sticks in each group as needed. Later, students use the bundles & sticks to develop and model strategies for double-digit addition and subtraction based on regrouping.




53 as 5 bundles of 10 and 3 singles

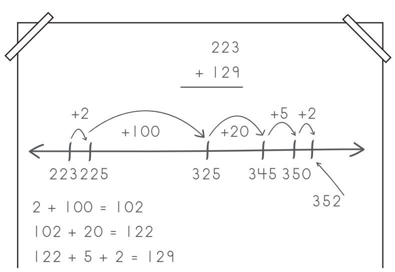
## Number Line

At first, the number line is used simply to represent numbers in order. As students become more comfortable with the number line, they use it to develop and model strategies for adding and subtracting. The open number line allows students to partition as they see fit; since they no longer need to count by 1s, first graders' number lines reflect a growing sense of magnitude and increasingly illustrate strategies that involve a flexible approach to working with ones and tens.

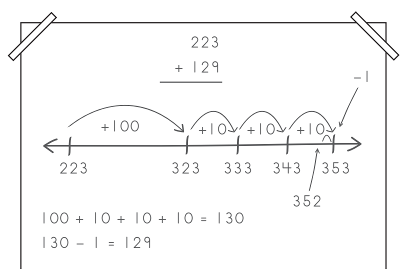



Jumping Strategy in Grade 1

Second and third graders use the number line to develop and model a variety of strategies for adding and subtracting multi-digit numbers, many of which do not involve regrouping. The open number line allows students to partition the number line as they see fit and to use landmark numbers that are comfortable for them.




In this example, the student began at 223 and added 129 in smaller pieces to get to the sum, 352.

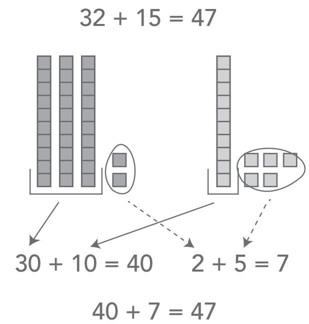



In this example, the student added 100 and 3 tens to 223. Then they subtracted 1 to compensate, for a total of 352.

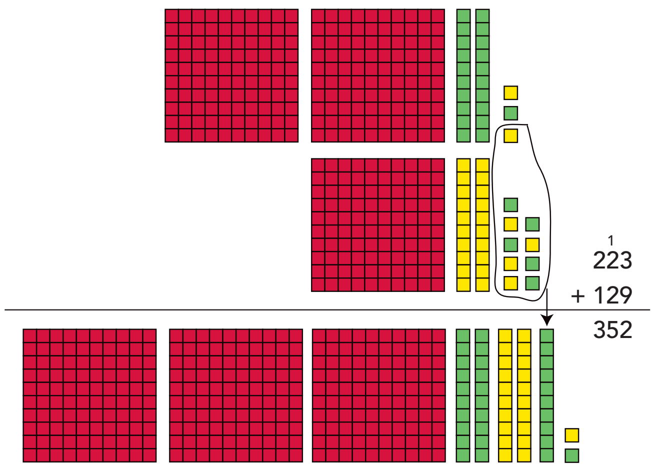
The open number line provides a way for students to develop strategies, keep track of the steps involved in each strategy, and communicate about their strategies with others.

## Base Ten Area Pieces

Starting in grade 2 and continuing through grade 4, students use base ten area pieces, which are pregrouped into pieces of 1, 10, and 100, to represent and compute with multi-digit numbers. With the bundles & sticks model, students are able to physically put together and take apart bundles of 10 sticks. Because the base ten area pieces are pregrouped, students cannot take them apart and assemble them: instead, they will need to trade a tens piece for 10 ones or a hundreds piece for 10 tens pieces, and so on. It is crucial for the development of place value understanding that students have many opportunities to group and ungroup and to trade pieces.



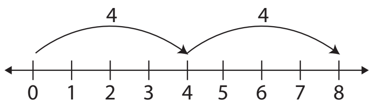

Base ten pieces play an important role in introducing the standard algorithms for addition and subtraction. Unlike the open number line, the pieces focus squarely on each place and value. So, for example, it is easy to see and understand that when you add the ones, you'll need to regroup (trade 10 ones for a ten or carry a ten to the tens place) if, after adding, you end up with more than 9 ones.



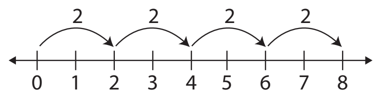
# MODELS FOR MULTIPLICATION & DIVISION TO 100

## Number Lines

Starting in grade 3, students also use the number line as a way to understand multiplication, modeling the repeated addition in multiplication situations and looking for patterns and relationships that emerge.




$2 \times 4 = 8$

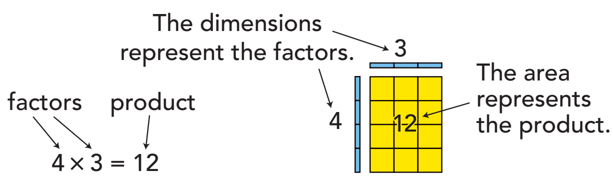



$4 \times 2 = 8$

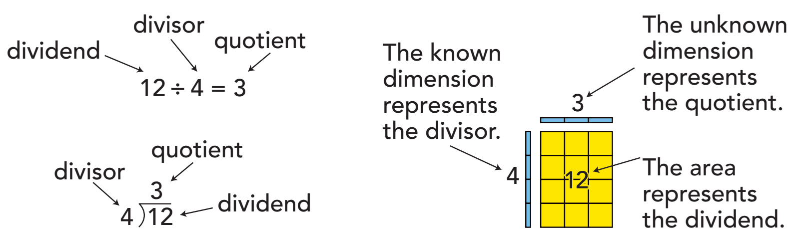
If you start at 0 and take 2 jumps of 4, you land on 8.
If you take 4 jumps of 2, you also land on 8.

## Array or Area Model

In the area model for multiplication, the total area of the rectangle represents the product, and the two dimensions represent the factors.




Because multiplication and division are inverse operations, the same model can be used to illustrate division.




## Ratio Tables

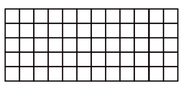
Starting in grade 3, the ratio table is used to simultaneously build multiplicative thinking and proportional reasoning. The model is introduced in Unit 1 to represent students' strategies. Students will fill in tables for situations with a constant ratio, such as when one row in a box has 8 crayons and there are 4 rows. Later, the ratio table will become a tool for students to use in problem-solving to compute multiplication, division, and fraction problems, as well as make conversions. This model will also be used for many years to come in higher mathematics to model proportional situations.

| Rows of crayons | Number of crayons |
|---|---|
| 1 | 8 |
| 2 | 16 |
| ×2 3 | 24 ×2 |
| 6 | 48 |

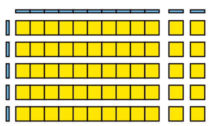
# MODELS FOR MULTI-DIGIT MULTIPLICATION & DIVISION

## Array or Area Model

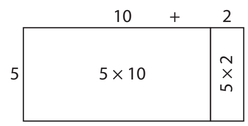
Bridges helps students use the array model for multiplication by beginning with discrete models in third grade. Students progress over time, using closed arrays, base ten area pieces with linear pieces, and then open arrays. With closed arrays, they can count each square unit by 1s. With base ten area pieces and linear pieces, the area is now modeled in bigger chunks, tens and ones, and the dimensions are defined with linear pieces, helping students differentiate between linear measures and area measures. With open arrays, students can chunk the arrays into pieces that are convenient and efficient for the problem. With each model, students can chunk areas into bigger pieces, moving away from counting strategies to repeated addition and then to multiplicative thinking.



Closed Tile Array



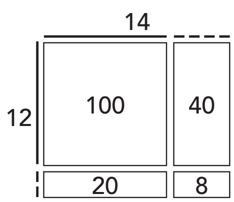
Closed Array Built with Base Ten Pieces




Open Array

While students will discover many ways to solve multiplication and division problems, the array model provides a way for them to discuss their strategies with one another, decompose the numbers, apply the distributive property, and identify partial products.

Multiplication

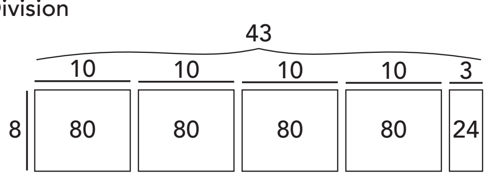



12 × 14 = 168

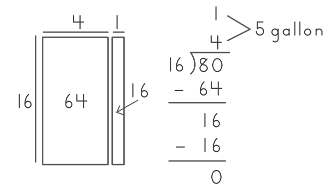
10 × 10 = 100
10 × 2 = 20
10 × 4 = 40
2 × 4 = 8
100 + 60 + 8 = 168

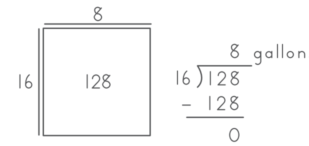
$$\begin{array}{r} 12 \\ \times\ 14 \\ \hline 48 \\ +\ 120 \\ \hline 168 \end{array}$$

Division




80 + 80 + 80 + 80 = 320
320 + 24 = 344
8×43 = 344
so 344 ÷ 8 = 43




The array model helps fifth graders develop increasingly efficient methods for building, sketching, and recording multi-digit multiplication and division problems. It also provides a conceptual foundation that allows students to use algorithms with understanding.

## Ratio Tables

In using ratio tables to solve division combinations, students place the divisor in the first row or column, then build up to the dividend by groups of the divisor, using the table to record their thinking. Ratio tables can be either vertical or horizontal, and you will find examples of both. Students will often write equations alongside their tables to clarify their thinking.

Strategy for solving 243 ÷ 9

20 + 5 + 2

| Number of Groups | 1 | 10 | 20 | 5 | 2 | 27 |
|---|---|---|---|---|---|---|
| Total | 9 | 90 | 180 | 45 | 18 | 243 |

180 + 45 + 18

Another strategy for solving 243 ÷ 9

| Groups | Total |
|---|---|
| 1 | 9 |
| 10 | 90 |
| 20 | 180 |
| 5 | 45 |
| 2 | 18 |
| 27 | 243 |

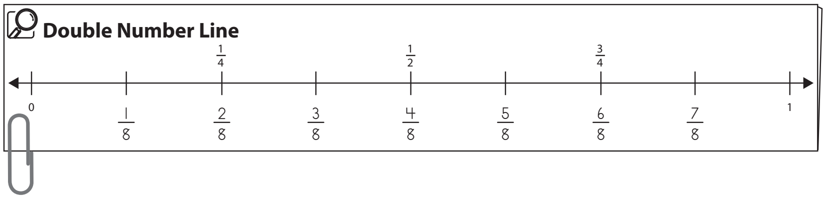
243 ÷ 9 = 27

1 × 9 = 9
10 × 9 = 90
20 × 9 = 180
5 × 9 = 45
2 × 9 = 18

# MODELS FOR FRACTIONS

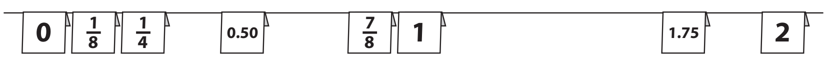
## Double Number Line

A special number line marked from 0 to 1 offers third graders a way to visualize fractional increments on this familiar yet powerful model. Just as the open number line allows students to visualize the relationships among multipliers, the special number line for fractions invites them to explore the relationships among different fractions. As they divide the number line into fractional increments, they find out where the positions of various fractions "line up."



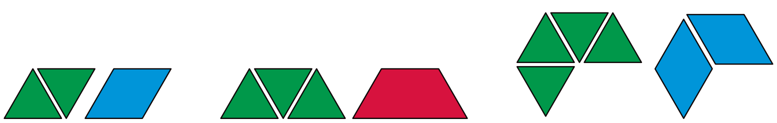

## Classroom Number Line

Students also use a number line model to compare and order fractions and decimals. By reasoning about the relationships, students place fractions and decimals on a class number line. For example, to place $\frac{7}{8}$ on the number line, students reason that $\frac{7}{8}$ is $\frac{1}{8}$ less than $\frac{8}{8} = 1$. Given the position of $\frac{1}{4}$, which is already placed on the number line, they find the position of $\frac{1}{8}$. Then they can find $\frac{7}{8}$ as $\frac{1}{8}$ to the left of 1.




## Pattern Blocks

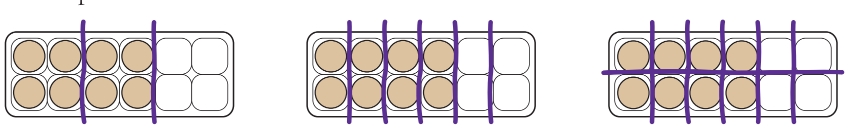
Pattern blocks and folded paper rectangles offer students a way to understand fractions. When the yellow hexagon is understood to represent 1 whole, students can model halves with red trapezoids, thirds with blue rhombuses, and sixths with green triangles. These models make it possible for students to compare fractions by size and to generate equivalent fractions with conceptual understanding.



$\frac{2}{6} = \frac{1}{3}$ $\frac{3}{6} = \frac{1}{2}$ $\frac{4}{6} = \frac{2}{3}$

## Egg Carton Model

The egg carton model, introduced in Grade 3 and used again in Grade 4, involves a 12-egg carton. While most visual or physical models of fractions specifically portray them as either parts of a whole or parts of a set, the egg carton simultaneously models fractions both ways. Eight eggs set into a carton of 12 can be viewed as $\frac{2}{3}$ of a whole (the whole being the entire egg carton) or 8 out of a set of 12 eggs. However, if pieces of yarn or string are used to divide the carton, those 8 eggs can be seen to fill $\frac{2}{3}$, $\frac{4}{6}$, or $\frac{8}{12}$ of the carton, depending on the number of parts into which the carton has been divided.

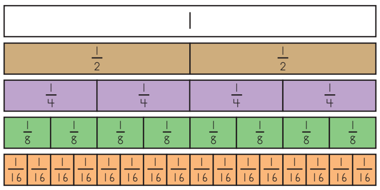


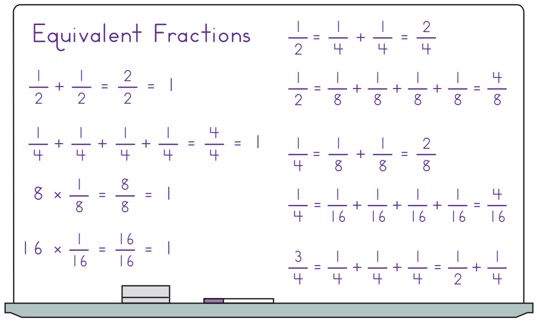
8 eggs fills $\frac{2}{3}$ of the carton | 8 eggs fills $\frac{4}{6}$ of the carton | 8 eggs fills $\frac{8}{12}$ of the carton

# MODELS FOR FRACTIONS & DECIMALS

## Folded Paper Strips

Students fold and then cut and label paper strips of different colors to form various fractions. These simple "fraction kits" make it easy for students to find many relationships among halves, fourths, eighths, and sixteenths, as well as recognize and generate equivalent fractions.

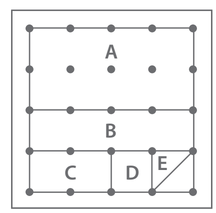



## Geoboard Fractions

Students also investigate fractions on the geoboard and make observations about a wide variety of fractional relationships. Much like the egg carton model, the geoboard can be used to characterize fractions as parts of a whole or parts of a set. The entire board is assigned a value of 1, but there are 16 smaller squares within the larger whole. This makes the geoboard an especially effective tool for recognizing and generating equivalent fractions, comparing fractions, composing and decomposing fractions, and multiplying fractions by whole numbers.

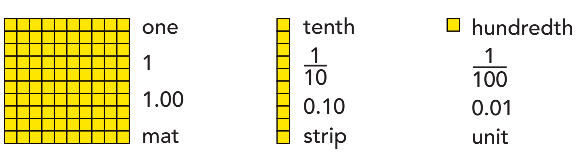
| Region A | Region B | Region C | Region D | Region E |
|---|---|---|---|---|
| ○ $\frac{1}{2}$ of a large square<br>○ 8 out of 16 little squares, or $\frac{8}{16}$<br>○ $\frac{8}{16}+\frac{8}{16}=\frac{16}{16}=1$<br>○ $2\times\frac{8}{16}=\frac{16}{16}=1$<br>○ region A equals 4 Cs $\frac{1}{2}=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}$ | ○ $\frac{1}{4}$ of a large square<br>○ 4 out of 16 little squares, or $\frac{4}{16}$<br>○ $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{4}{16}$<br>○ $4\times\frac{1}{16}=\frac{4}{16}$<br>○ region B equals 8 Es $\frac{1}{4}=8\times\frac{1}{32}$ | ○ $\frac{1}{8}$ of a large square<br>○ 2 out of 16 little squares, or $\frac{2}{16}$<br>○ $\frac{1}{16}+\frac{1}{16}=\frac{2}{16}$<br>○ $2\times\frac{1}{16}=\frac{2}{16}$<br>○ region C is half of B $\frac{1}{8}+\frac{1}{8}=\frac{2}{8}=\frac{1}{4}$<br>○ region C is twice of D $\frac{1}{4}=2\times\frac{1}{8}$ | ○ $\frac{1}{16}$ of a large square<br>○ 1 out of 16 little squares<br>○ region B equals 4 Ds $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{4}{16}=\frac{1}{4}$<br>○ region C equals 2 Ds $\frac{1}{16}+\frac{1}{16}=\frac{2}{16}=\frac{1}{8}$<br>○ $16\times\frac{1}{16}=1$ | ○ $\frac{1}{32}$ of a large square<br>○ $\frac{1}{2}$ of one of the little squares<br>○ 4 Es makes one C $\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}=\frac{4}{32}=\frac{1}{8}$<br>○ $32\times\frac{1}{32}=1$ |




If the whole geoboard has an area of 1, what fraction of the board does each region show, and how do those regions relate to one another?

## Base Ten Pieces

The base ten pieces, used extensively for modeling and computing with whole numbers, are also a very effective tool for modeling and comparing decimal fractions and numbers when the mat is assigned a value of 1.



# MODELS FOR ADDING & SUBTRACTING FRACTIONS

## Money

The use of money enables students to build intuitions about common denominators in a relatively painless way because most fifth graders are accustomed to thinking of coins and combinations of coins in terms of pennies, and most are aware of the relationship between coins and fractions (a quarter is $\frac{1}{4}$ of a dollar or $\frac{25}{100}$; a dime is $\frac{1}{10}$ of a dollar or $\frac{10}{100}$, and so on). In relating fractions and money, students might approach the task of adding $\frac{3}{10}$ and $\frac{7}{20}$ by thinking of adding 3 dimes and 7 nickels, or 30¢ + 35¢. The answer is 65¢ or \$0.65 or $\frac{65}{100}$ of a dollar, the conversion to like denominators (hundredths or pennies) having taken place at an almost invisible level.

$$\frac{3}{10} + \frac{7}{20} = \$0.30 + \$0.35 = \$0.65 = \frac{65}{100} = \frac{13 \text{ nickels}}{20 \text{ nickels}} = \frac{13}{20}$$

**Money Model**

## Time

Like money, time enables students to build intuitions about common denominators, because any fraction of an hour can be thought of in terms of minutes. For example, adding $\frac{5}{6}$ and $\frac{1}{12}$ of an hour can be thought of as adding 50 minutes and 5 minutes, or $\frac{50}{60} + \frac{5}{60}$. The total is 55 minutes, which can also be expressed as $\frac{55}{60}$. If one thinks of the two addends, $\frac{5}{6}$ and $\frac{1}{12}$, in terms of 5-minute chunks, or twelfths, the combination can also be expressed as $\frac{10}{12} + \frac{1}{12}$, or $\frac{11}{12}$. Here again, the act of rewriting fractions so they have like denominators takes place in a context that's familiar and fairly intuitive for many students.

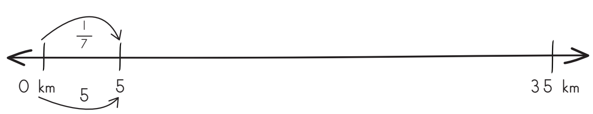
$$\frac{5}{6} + \frac{1}{12} \qquad \frac{50 \text{ minutes}}{60 \text{ minutes}} + \frac{5 \text{ minutes}}{60 \text{ minutes}} = \frac{55 \text{ minutes}}{60 \text{ minutes}} = \frac{11 \text{ sets of 5 minutes}}{12 \text{ sets of 5 minutes}} = \frac{11}{12}$$

**Clock Model**

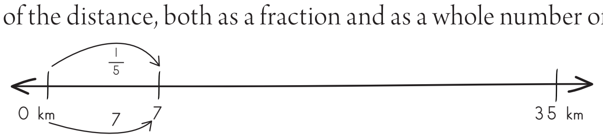
## Double Number Line

Many fraction combinations don't lend themselves to thinking in terms of money or time, which work only if both denominators are either factors of 100 or factors of 60. A combination such as $\frac{1}{7} + \frac{3}{5}$ is one of these. To deal with such situations, students are introduced to the double number line model, presented in the context of a running trail. Given a situation in which Mr. Miles ran $\frac{1}{7}$ of the trail and then ran $\frac{3}{5}$ of it, and he wants to determine the fraction of the trail he covered in all, students are asked to think of a trail length that is a multiple of both 7 and 5. A common response is 35 kilometers. Once the length has been selected, students employ the following steps to model and solve the problem:

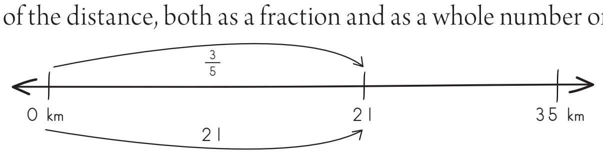
» Draw and label a number line with 0 at one end and 35 at the other.




» Find and mark $\frac{1}{7}$ of the distance, both as a fraction and as a whole number of kilometers.

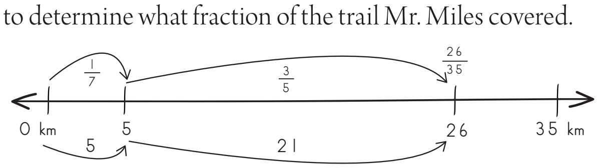



» Find and mark $\frac{1}{5}$ of the distance, both as a fraction and as a whole number of kilometers.




» Scale up to find and mark $\frac{3}{5}$ of the distance.

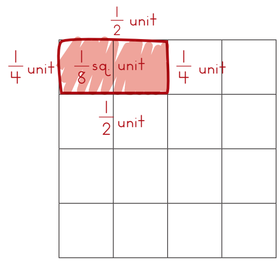
» Add the distances to determine what fraction of the trail Mr. Miles covered.



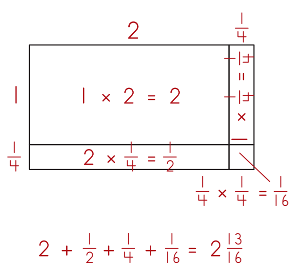
# MODELS FOR MULTIPLYING & DIVIDING FRACTIONS & DECIMALS

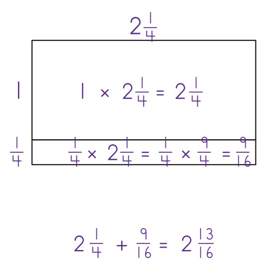
## Area Model

When students multiply fractions without a visual model and without considering units, they often find it confusing that the product of two fractions is smaller than each of the two fractions being multiplied. For example, the product of ½ and ¼ is ⅛. However, this fact makes much more sense when students consider the two fractions and their product using the area model: each dimension is a fraction of a single linear unit, and the area of the resulting rectangle is an even smaller fraction of the entire square unit



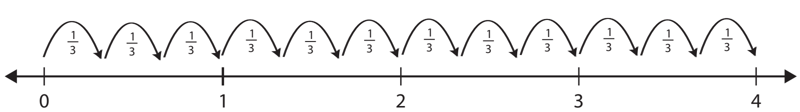

The area model offers a powerful tool for showing and solving problems that involve multiplying mixed numbers. It provides a structure for multiplying mixed numbers and connects multiplying mixed numbers to students' work with multiplying whole numbers. Splitting mixed numbers into whole numbers and fractions generates four manageable multiplication problems. As students become more efficient, some will split the array into two parts instead of four. This foundation will help them as they move on to more abstract strategies.




## Number Line

A problem like 4 ÷ ⅓ might be posed in the context of a situation that invites students to think about how many times you have to iterate ⅓ to make 4. They can use the number line to make equal jumps of ⅓ from 0 to 4. Because there are 3 jumps of ½ between each pair of whole numbers, there are 12 thirds in 4, so 4 ÷ ⅓ = 12.




## Ratio Tables

To find 19 × 2.25, students start with one set of 2.25 and multiply up from there. This ratio table shows an over strategy.